

# On central loops and the central square property <sup>\*†</sup>

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## Abstract

The representation sets of a central square C-loop are investigated. Isotopes of central square C-loops of exponent 4 are shown to be both C-loops and A-loops.

## 1 Introduction

C-loops are one of the least studied loops. Few publications that have considered C-loops include Fenyves [10], [11], Beg [3], [4], Phillips et. al. [17], [19], [15], [14], Chein [7] and Solarin et. al. [2], [23], [21], [20]. The difficulty in studying them is as a result of the nature of their identities when compared with other Bol-Moufang identities (the element occurring twice on both sides has no other element separating it from itself). Latest publications on the study of C-loops which has attracted fresh interest on the structure include [17], [19], and [15].

LC-loops, RC-loops and C-loops are loops that satisfies the identities  $(xx)(yz) = (x(xy))z$ ,  $(zy)(xx) = z((yx)x)$  and  $x(y(yz)) = ((xy)y)z$  respectively. Fenyves' work in [11] was completed in [17]. Fenyves proved that LC-loops and RC-loops are defined by three equivalent identities. But in [17] and [18], it was shown that LC-loops and RC-loops are defined by four equivalent identities. Solarin [21] named the fourth identities left middle(LM-) and right middle(RM-) identities and loops that obey them are called LM-loops and RM-loops respectively. These terminologies were also used in [22]. Their basic properties are found in [19], [11] and [9].

**Definition 1.1** *A set  $\Pi$  of permutations on a set  $L$  is the representation of a loop  $(L, \cdot)$  if and only if*

- (i)  $I \in \Pi$  (identity mapping),
- (ii)  $\Pi$  is transitive on  $L$  (i.e for all  $x, y \in L$ , there exists a unique  $\pi \in \Pi$  such that  $x\pi = y$ ),

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(iii) if  $\alpha, \beta \in \Pi$  and  $\alpha\beta^{-1}$  fixes one element of  $L$ , then  $\alpha = \beta$ .

The left(right) representation of a loop  $L$  is denoted by  $\Pi_\lambda(L)(\Pi_\rho(L))$  or  $\Pi_\lambda(\Pi_\rho)$  and is defined as the set of all left(right) translation maps on the loop i.e if  $L$  is a loop, then

$\Pi_\lambda = \{L_x : L \rightarrow L \mid x \in L\}$  and  $\Pi_\rho = \{R_x : L \rightarrow L \mid x \in L\}$  where  $R_x : L \rightarrow L$  and

$L_x : L \rightarrow L$  defined as  $yR_x = yx$  and  $yL_x = xy$  respectively for all  $x, y \in L$  are bijections.

**Definition 1.2** Let  $(L, \cdot)$  be a loop. The left nucleus of  $L$  is the set

$$N_\lambda(L, \cdot) = \{a \in L : ax \cdot y = a \cdot xy \ \forall x, y \in L\}.$$

The right nucleus of  $L$  is the set

$$N_\rho(L, \cdot) = \{a \in L : y \cdot xa = yx \cdot a \ \forall x, y \in L\}.$$

The middle nucleus of  $L$  is the set

$$N_\mu(L, \cdot) = \{a \in L : ya \cdot x = y \cdot ax \ \forall x, y \in L\}.$$

The nucleus of  $L$  is the set

$$N(L, \cdot) = N_\lambda(L, \cdot) \cap N_\rho(L, \cdot) \cap N_\mu(L, \cdot).$$

The centrum of  $L$  is the set

$$C(L, \cdot) = \{a \in L : ax = xa \ \forall x \in L\}.$$

The center of  $L$  is the set

$$Z(L, \cdot) = N(L, \cdot) \cap C(L, \cdot).$$

$L$  is said to be a centrum square loop if  $x^2 \in C(L, \cdot)$  for all  $x \in L$ .  $L$  is said to be a central square loop if  $x^2 \in Z(L, \cdot)$  for all  $x \in L$ .  $L$  is said to be left alternative if for all  $x, y \in L$ ,  $x \cdot xy = x^2y$  and is said to right alternative if for all  $x, y \in L$ ,  $yx \cdot x = yx^2$ . Thus,  $L$  is said to be alternative if it is both left and right alternative. The triple  $(U, V, W)$  such that  $U, V, W \in \text{SYM}(L, \cdot)$  is called an autotopism of  $L$  if and only if

$$xU \cdot yV = (x \cdot y)W \ \forall x, y \in L.$$

$\text{SYM}(L, \cdot)$  is called the permutation group of the loop  $(L, \cdot)$ . The group of autotopisms of  $L$  is denoted by  $\text{AUT}(L, \cdot)$ . Let  $(L, \cdot)$  and  $(G, \circ)$  be two distinct loops. The triple  $(U, V, W) : (L, \cdot) \rightarrow (G, \circ)$  such that  $U, V, W : L \rightarrow G$  are bijections is called a loop isotopism if and only if

$$xU \circ yV = (x \cdot y)W \ \forall x, y \in L.$$

In [13], the three identities stated in [11] were used to study finite central loops and the isotopes of central loops. It was shown that in a finite RC(LC)-loop  $L$ ,  $\alpha\beta^2 \in \Pi_\rho(L)(\Pi_\lambda(L))$

for all  $\alpha, \beta \in \Pi_\rho(L)(\Pi_\lambda(L))$  while in a C-loop  $L$ ,  $\alpha^2\beta \in \Pi_\rho(L)(\Pi_\lambda(L))$  for all  $\alpha, \beta \in \Pi_\rho(L)(\Pi_\lambda(L))$ . A C-loop is both an LC-loop and an RC-loop ([11]), hence it satisfies the formal. Here, it will be shown that LC-loops and RC-loops satisfy the later formula.

Also in [13], under a triple of the form  $(A, B, B)((A, B, A))$ , alternative centrum square loop isotopes of centrum square C-loops were shown to be C-loops. It will be shown here that the same result is true for RC(LC)-loops.

It is shown that a finite loop is a central square central loop if and only if its left and right representations are closed relative to some left and right translations.

Central square C-loops of exponent 4 are shown to be groups, hence their isotopes are both C-loops and A-loops.

For definition of concepts in theory of loops readers may consult [5], [22] and [16].

## 2 Preliminaries

**Definition 2.1** ([16]) Let  $(L, \cdot)$  be a loop and  $U, V, W \in \text{SYM}(L, \cdot)$ .

1. If  $(U, V, W) \in \text{AUT}(L, \cdot)$  for some  $U, V, W$ , then  $U$  is called an autotopic,
  - the set of autotopic bijections in a loop  $(L, \cdot)$  is represented by  $\Sigma(L, \cdot)$ .
2. If there exists  $V \in \text{SYM}(L, \cdot)$  such that  $xU \cdot y = x \cdot yV$  for all  $x, y \in L$ , then  $U$  is called  $\mu$ -regular while  $U' = V$  is called its adjoint.
  - The set of all  $\mu$ -regular bijections in a loop  $(L, \cdot)$  is denoted by  $\Phi(L, \cdot)$ , while the collection of all adjoints in the loop  $(L, \cdot)$  is denoted by  $\Phi^*(L, \cdot)$ .

**Theorem 2.1** ([16]) If two quasigroups are isotopic then their groups of autotopisms are isomorphic.

**Theorem 2.2** ([16]) The set  $\Phi(Q, \cdot)$  of all  $\mu$ -regular bijections of a quasigroup  $(Q, \cdot)$  is a subgroup of the group  $\Sigma(Q, \cdot)$  of all autotopic bijections of  $(Q, \cdot)$ .

**Corollary 2.1** ([16]) If two quasigroups  $Q$  and  $Q'$  are isotopic, then the corresponding groups  $\Phi$  and  $\Phi'$  [ $\Phi^*$  and  $\Phi'^*$ ] are isomorphic.

**Definition 2.2** A loop  $(L, \cdot)$  is called a left inverse property loop or right inverse property loop (L.I.P.L. or R.I.P.L.) if and only if it obeys the left inverse property or right inverse property (L.I.P or R.I.P):  $x^\lambda(xy) = y$  or  $(yx)x^\rho = y$ . Hence, it is called an inverse property loop (I.P.L.) if and only if it has the inverse property (I.P.) i.e. it is both an L.I.P. and an R.I.P. loop.

Most of our results and proofs, are stated and written in dual form relative to RC-loops and LC-loops. That is, a statement like 'LC(RC)-loop... A(B)' where 'A' and 'B' are some equations or expressions simply means 'A' is for LC-loops while 'B' is for RC-loops. This is done so that results on LC-loops and RC-loops can be combined to derive those on C-loops. For instance an LC(RC)-loop is a L.I.P.L.(R.I.P.L) loop while a C-loop in an I.P.L. loop.

### 3 Finite Central Loops

**Lemma 3.1** *Let  $L$  be a loop.  $L$  is an  $LC(RC)$ -loop if and only if  $\beta \in \Pi_\rho(\Pi_\lambda)$  implies  $\alpha\beta \in \Pi_\rho(\Pi_\lambda)$  for some  $\alpha \in \Pi_\rho(\Pi_\lambda)$ .*

**Proof**

$L$  is an  $LC$ -loop if and only if  $x \cdot (y \cdot yz) = (x \cdot yy)z$  for all  $x, y, z \in L$  while  $L$  is an  $RC$ -loop if and only if  $(zy \cdot y)x = z(yy \cdot x)$  for all  $x, y, z \in L$ . Thus,  $L$  is an  $LC$ -loop if and only if  $xR_{y \cdot yz} = xR_{y^2}R_z$  if and only if  $R_{y^2}R_z = R_{y \cdot yz}$  for all  $y, z \in L$  and  $L$  is an  $RC$ -loop if and only if  $xL_{zy \cdot y} = xL_{y^2}L_z$  if and only if  $L_{zy \cdot y} = L_{y^2}L_z$ . With  $\alpha = R_{y^2}(L_{y^2})$  and  $\beta = R_z(L_z)$ ,  $\alpha\beta \in \Pi_\rho(\Pi_\lambda)$ . The converse is achieved by reversing the process. ♠

**Lemma 3.2** *A loop  $L$  is an  $LC(RC)$ -loop if and only if  $\alpha^2\beta = \beta\alpha^2$  for all  $\alpha \in \Pi_\lambda(\Pi_\rho)$  and  $\beta \in \Pi_\rho(\Pi_\lambda)$ .*

**Proof**

$L$  is an  $LC$ -loop if and only if  $x(x \cdot yz) = (x \cdot xy)z$  while  $L$  is an  $RC$ -loop if and only if  $(zy \cdot x)x = z(yx \cdot x)$ . Thus, when  $L$  is an  $LC$ -loop,  $yR_zL_x^2 = yL_x^2R_z$  if and only if  $R_zL_x^2 = L_x^2R_z$  while when  $L$  is an  $RC$ -loop,  $yL_zR_x^2 = yR_x^2L_z$  if and only if  $L_zR_x^2 = R_x^2L_z$ . Thus, replacing  $L_x(R_x)$  and  $R_z(L_z)$  respectively with  $\alpha$  and  $\beta$ , the result follows. The converse is achieved by doing the reverse. ♠

**Theorem 3.1** *Let  $L$  be a loop.  $L$  is an  $LC(RC)$ -loop if and only if  $\alpha, \beta \in \Pi_\lambda(\Pi_\rho)$  implies  $\alpha^2\beta \in \Pi_\lambda(\Pi_\rho)$ .*

**Proof**

$L$  is an  $LC$ -loop if and only if  $x \cdot (y \cdot yz) = (x \cdot yy)z$  for all  $x, y, z \in L$  while  $L$  is an  $RC$ -loop if and only if  $(zy \cdot y)x = z(yy \cdot x)$  for all  $x, y, z \in L$ . Thus when  $L$  is an  $LC$ -loop,  $zL_{x \cdot yy} = zL_y^2L_x$  if and only if  $L_y^2L_x = L_{x \cdot yy}$  while when  $L$  is an  $RC$ -loop,  $zR_y^2R_x = zR_{yy \cdot x}$  if and only if  $R_y^2R_x = R_{yy \cdot x}$ . Replacing  $L_y(R_y)$  and  $L_x(R_x)$  with  $\alpha$  and  $\beta$  respectively, we have  $\alpha^2\beta \in \Pi_\lambda(\Pi_\rho)$  when  $L$  is an  $LC(RC)$ -loop. The converse follows by reversing the procedure. ♠

**Theorem 3.2** *Let  $L$  be an  $LC(RC)$ -loop.  $L$  is centrum square if and only if  $\alpha \in \Pi_\rho(\Pi_\lambda)$  implies  $\alpha\beta \in \Pi_\rho(\Pi_\lambda)$  for some  $\beta \in \Pi_\rho(\Pi_\lambda)$ .*

**Proof**

By Lemma 3.1,  $R_{y^2}R_z = R_{y \cdot yz}(L_{y^2}L_z = L_{zy \cdot y})$ . Using Lemma 3.2, if  $L$  is centrum square,  $R_{y^2} = L_{y^2}(L_y^2 = R_{y^2})$ . So: when  $L$  is an  $LC$ -loop,  $R_{y^2}R_z = L_y^2R_z = R_zL_y^2 = R_zR_{y^2} = R_{y \cdot yz}$  while when  $L$  is an  $RC$ -loop,  $L_{y^2}L_z = R_y^2L_z = L_zR_{y^2} = L_zL_{y^2} = L_{zy \cdot y}$ . Let  $\alpha = R_z(L_z)$  and  $\beta = R_{y^2}(L_{y^2})$ , then  $\alpha\beta \in \Pi_\rho(\Pi_\lambda)$  for some  $\beta \in \Pi_\rho(\Pi_\lambda)$ .

Conversely, if  $\alpha\beta \in \Pi_\rho(\Pi_\lambda)$  for some  $\beta \in \Pi_\rho(\Pi_\lambda)$  such that  $\alpha = R_z(L_z)$  and  $\beta = R_{y^2}(L_{y^2})$  then  $R_zR_{y^2} = R_{y \cdot yz}(L_zL_{y^2} = L_{zy \cdot y})$ . By Lemma 3.1,  $R_{y^2}R_z = R_{y \cdot yz}(L_{zy \cdot y} = L_{y^2}L_z)$ , thus  $R_zR_{y^2} = R_{y^2}R_z(L_zL_{y^2} = L_{y^2}L_z)$  if and only if  $xz \cdot y^2 = xy^2 \cdot z(y^2 \cdot zx = z \cdot y^2x)$ . Let  $x = e$ , then  $zy^2 = y^2z(y^2z = zy^2)$  implies  $L$  is centrum square. ♠

**Corollary 3.1** *Let  $L$  be a loop.  $L$  is a centrum square  $LC(RC)$ -loop if and only if*

1.  $\alpha\beta \in \Pi_\rho(\Pi_\lambda)$  for all  $\alpha \in \Pi_\rho(\Pi_\lambda)$  and for some  $\beta \in \Pi_\rho(\Pi_\lambda)$ ,
2.  $\alpha\beta \in \Pi_\rho(\Pi_\lambda)$  for all  $\beta \in \Pi_\rho(\Pi_\lambda)$  and for some  $\alpha \in \Pi_\rho(\Pi_\lambda)$ .

**Proof**

This follows from Lemma 3.1 and Theorem 3.2.♠

## 4 Isotope of Central Loops

It must be mentioned that central loops are not conjugacy closed loops(CC-loops) as concluded in [23] or else a study of the isotopic invariance of C-loops will be trivial. This is because if C-loops are CC-loops, then a commutative C-loop would be a group since commutative CC-loops are groups. But from the constructions in [19], there are commutative C-loops that are not groups. The conclusion in [23] is based on the fact that the authors considered a loop of units in a central Algebra. This has also been observed in [1].

**Theorem 4.1** *Let  $(L, \cdot)$  be a loop.  $L$  is an  $LC(RC)$ -loop if and only if  $(R_{y^2}, L_y^{-2}, I)((R_y^2, L_{y^2}^{-1}, I)) \in AUT(L, \cdot)$  for all  $y \in L$ .*

**Proof**

According to [19],  $L$  is an LC-loop if and only if  $x \cdot (y \cdot yz) = (x \cdot yy)z$  for all  $x, y, z \in L$  while  $L$  is an RC-loop if and only if  $(zy \cdot y)x = z(yy \cdot x)$  for all  $x, y, z \in L$ .  $x \cdot (y \cdot yz) = (x \cdot yy)z$  if and only if  $x \cdot zL_y^2 = xR_{y^2} \cdot z$  if and only if  $(R_{y^2}, L_y^{-2}, I) \in AUT(L, \cdot)$  for all  $y \in L$  while  $(zy \cdot y)x = z(yy \cdot x)$  if and only if  $zR^2 \cdot x = z \cdot xL_{y^2}$  if and only if  $(R_y^2, L_{y^2}^{-1}, I) \in AUT(L, \cdot)$  for all  $y \in L$ .♠

**Corollary 4.1** *Let  $(L, \cdot)$  be an  $LC(RC)$ -loop,  $(R_{y^2}L_x^2, L_y^{-2}, L_x^2)((R_y^2, L_{y^2}^{-1}R_x^2, R_x^2)) \in AUT(L, \cdot)$  for all  $x, y \in L$ .*

**Proof**

In an LC-loop  $L$ ,  $(L_x^2, I, L_x^2) \in AUT(L, \cdot)$  while in an RC-loop  $L$ ,  $(I, R_x^2, R_x^2) \in AUT(L, \cdot)$ . Thus by Theorem 4.1 : for an LC-loop,  $(R_{y^2}, L_y^{-2}, I)(L_x^2, I, L_x^2) = (R_{y^2}L_x^2, L_y^{-2}, L_x^2) \in AUT(L, \cdot)$  and for an RC-loop,  $(R_y^2, L_{y^2}^{-1}, I)(I, R_x^2, R_x^2) = (R_y^2, L_{y^2}^{-1}R_x^2, R_x^2) \in AUT(L, \cdot)$ .♠

**Theorem 4.2** *Let  $(L, \cdot)$  be a loop.  $L$  is a C-loop if and only if  $L$  is a right (left) alternative  $LC(RC)$ -loop.*

**Proof**

If  $(L, \cdot)$  is an  $LC(RC)$ -loop, then by Theorem 4.1,  $(R_{y^2}, L_y^{-2}, I)((R_y^2, L_{y^2}^{-1}, I)) \in AUT(L, \cdot)$  for all  $y \in L$ . If  $L$  has the right(left) alternative property,  $(R_y^2, L_y^{-2}, I) \in AUT(L, \cdot)$  for all  $y \in L$  if and only if  $L$  is a C-loop.♠

**Lemma 4.1** *Let  $(L, \cdot)$  be a loop.  $L$  is an  $LC(RC, C)$ -loop if and only if  $R_{y^2}(R_y^2, R_y^2) \in \Phi(L)$  and  $(R_{y^2})^* = L_y^2((R_y^2)^* = L_{y^2}, (R_y^2)^* = L_y^2) \in \Phi^*(L)$  for all  $y \in L$ .*

**Proof**

This can be interpreted from Theorem 4.1.♠

**Theorem 4.3** *Let  $(G, \cdot)$  and  $(H, \circ)$  be two distinct loops. If  $G$  is a central square  $LC(RC)$ -loop,  $H$  an alternative central square loop and the triple  $\alpha = (A, B, B)$  ( $\alpha = (A, B, A)$ ) is an isotopism of  $G$  upon  $H$ , then  $H$  is a  $C$ -loop.*

**Proof**

$G$  is a  $LC(RC)$ -loop if and only if  $R_{y^2}(R_y^2) \in \Phi(G)$  and  $(R_{y^2})^* = L_y^2((R_y^2)^* = L_{y^2}) \in \Phi^*(G)$  for all  $x \in G$ . Using the idea in [6] :  $L'_{xA} = B^{-1}L_xB$  and  $R'_{xB} = A^{-1}R_xA$  for all  $x \in G$ . Using Corollary 2.1, for the case of  $G$  been an  $LC$ -loop : let  $h : \Phi(G) \rightarrow \Phi(H)$  and  $h^* : \Phi^*(G) \rightarrow \Phi^*(H)$  be defined as  $h(U) = B^{-1}UB \forall U \in \Phi(G)$  and  $h^*(V) = B^{-1}VB \forall V \in \Phi^*(G)$ . This mappings are isomorphisms. Using the hypothesis,  $h(R_{y^2}) = h(L_{y^2}) = h(L_y^2) = B^{-1}L_y^2B = B^{-1}L_yBB^{-1}L_yB = L'_{yA}L'_{yA} = L'^2_{yA} = L'_{(yA)^2} = R'_{(yA)^2} = R'^2_{(yA)} \in \Phi(H)$ .  $h^*[(R_{y^2})^*] = h^*(L_y^2) = B^{-1}L_y^2B = B^{-1}L_yL_yB = B^{-1}L_yBB^{-1}L_yB = L'_{yA}L'_{yA} = L'^2_{yA} \in \Phi^*(H)$ . So,  $R'^2_{yA} \in \Phi(H)$  and  $(R'^2_{yA})^* = L'^2_{yA} \in \Phi^*(H)$  for all  $y \in H$  if and only if  $H$  is a  $C$ -loop.

For the case of an  $RC$ -loop  $G$ , using  $h$  and  $h^*$  as above but now defined as :  $h(U) = A^{-1}UA \forall U \in \Phi(G)$  and  $h^*(V) = A^{-1}VA \forall V \in \Phi^*(G)$ . This mappings are still isomorphisms. Using the hypotheses,  $h(R_y^2) = A^{-1}R_y^2A = A^{-1}R_yAA^{-1}R_yA = R'_{yB}R'_{yB} = R'^2_{yB} \in \Phi(H)$ .  $h^*[(R_y^2)^*] = h^*(L_{y^2}) = h^*(R_{y^2}) = A^{-1}R_y^2A = A^{-1}R_yR_yB = B^{-1}R_yBB^{-1}R_yB = R'_{yA}R'_{yA} = R'^2_{yA} = R'_{(yA)^2} = L'_{(yA)^2} = L'^2_{yA} \in \Phi^*(H)$ . So,  $R'^2_{yA} \in \Phi(H)$  and  $(R'^2_{yA})^* = L'^2_{yA} \in \Phi^*(H)$  if and only if  $H$  is a  $C$ -loop.♠

**Corollary 4.2** *Let  $(G, \cdot)$  and  $(H, \circ)$  be two distinct loops. If  $G$  is a central square left (right)  $RC(LC)$ -loop,  $H$  an alternative central square loop and the triple  $\alpha = (A, B, B)$  ( $\alpha = (A, B, A)$ ) is an isotopism of  $G$  upon  $H$ , then  $H$  is a  $C$ -loop.*

**Proof**

By Theorem 4.2,  $G$  is a  $C$ -loop in each case. The rest of the proof follows by Theorem 4.3.♠

**Remark 4.1** *Corollary 4.2 is exactly what was proved in [13].*

## 5 Central square $C$ -loops of exponent 4

For a loop  $(L, \cdot)$ , the bijection  $J : L \rightarrow L$  is defined by  $xJ = x^{-1}$  for all  $x \in L$ .

**Theorem 5.1** *In a  $C$ -loop  $(L, \cdot)$ , if any of the following is true for all  $z \in L$ :*

1.  $(I, L_z^2, JL_z^2J) \in \text{AUT}(L)$ ,
2.  $(R_z^2, I, JR_z^2J) \in \text{AUT}(L)$ ,

then,  $L$  is a loop of exponent 4.

**Proof**

1. If  $(I, L_z^2, JL_z^2J) \in AUT(L)$  for all  $z \in L$ , then :  $x \cdot yL_z^2 = (xy)JL_z^2J$  for all  $x, y, z \in L$  implies  $x \cdot z^2y = xy \cdot z^{-2}$  implies  $z^2y \cdot z^2 = y$ . Then  $y^4 = e$ . Hence  $L$  is a C-loop of exponent 4.
2. If  $(R_z^2, I, JR_z^2J) \in AUT(L)$  for all  $z \in L$ , then :  $xR_z^2 \cdot y = (xy)JR_z^2J$  for all  $x, y, z \in L$  implies  $(xz^2) \cdot y = [(xy)^{-1}z^2]^{-1}$  implies  $(xz^2) \cdot y = z^{-2}(xy)$  implies  $(xz^2) \cdot y = z^{-2}x \cdot y$  implies  $xz^2 = z^{-2}x$  implies  $z^4 = e$ . Hence  $L$  is a C-loop of exponent 4.



**Theorem 5.2** *In a C-loop  $L$ , if the following are true for all  $z \in L$  :*

1.  $(I, L_z^2, JL_z^2J) \in AUT(L)$ ,
2.  $(R_z^2, I, JR_z^2J) \in AUT(L)$ ,

*then,  $L$  is a central square C-loop of exponent 4.*

**Proof**

By the first hypothesis, If  $(I, L_z^2, JL_z^2J) \in AUT(L)$  for all  $z \in L$ , then :  $x \cdot yL_z^2 = (xy)JL_z^2J$  for all  $x, y, z \in L$  implies  $x \cdot z^2y = xy \cdot z^{-2}$ .

By the second hypothesis, If  $(R_z^2, I, JR_z^2J) \in AUT(L)$  for all  $z \in L$ , then :  $xR_z^2 \cdot y = (xy)JR_z^2J$  for all  $x, y, z \in L$  implies  $xz^2 \cdot y = z^{-2}(xy)$ .

Using the two results above and keeping in mind that  $L$  is a C-loop we have :

$x \cdot z^2y = xz^2 \cdot y$  if and only if  $xy \cdot z^{-2} = z^{-2} \cdot xy$ . Let  $t = xy$  then  $tz^{-2} = z^{-2}t$  if and only if  $z^2t^{-1} = t^{-1}z^2$ . Let  $s = t^{-1}$  then  $z^2 \in C(L, \cdot)$  for all  $z \in L$ .

Since  $s$  is arbitrary in  $L$ , then the last result shows that  $L$  is centrum square. Furthermore, C-loops have been found to be nuclear square in [19], thus  $z^2 \in Z(L, \cdot)$ . Hence  $L$  is a central square C-loop. Finally, by Theorem 5.1,  $x^4 = e$ . ♠

**Remark 5.1** *In [19], C-loops of exponent 2 were found. But in this section we have further checked for the existence of C-loops of exponent 4 (Theorem 5.1). Also, in [19] and [11], C-loops are proved to be naturally nuclear square. Theorem 5.2 gives some conditions under which a C-loop can be naturally central square.*

**Theorem 5.3** *If  $A = (U, V, W) \in AUT(L, \cdot)$  for a C-loop  $(L, \cdot)$ , then  $A_\rho = (V, U, JWJ) \notin AUT(L, \cdot)$ , but  $A_\mu = (W, JVJ, U)$ ,  $A_\lambda = (JUJ, W, V) \in AUT(L, \cdot)$ .*

**Proof**

The fact that  $A_\mu, A_\lambda \in AUT(L, \cdot)$  has been shown in [5] and [16] for an I. P. L.  $L$ . Let  $L$  be a C-loop. Since C-loops are inverse property loops,  $A_\mu = (W, JVJ, U)$ ,  $A_\lambda = (JUJ, W, V) \in$

$AUT(L, \cdot)$ . A C-loop is both an RC-loop and an LC-loop. So,  $(I, R_x^2, R_x^2), (L_x^2, I, L_x^2) \in AUT(L, \cdot)$  for all  $x \in L$ . Thus, if  $A_\rho \in AUT(L, \cdot)$  when  $A = (I, R_x^2, R_x^2)$  and  $A = (L_x^2, I, L_x^2)$ ,  $A_\rho = (I, L_x^2, JL_x^2J) \in AUT(L)$  and  $A_\rho = (R_x^2, I, JR_x^2J) \in AUT(L)$  hence by Theorem 5.1 and Theorem 5.2, all C-loops are central square and of exponent 4 (in fact it will soon be seen in Theorem 5.4 that central square C-loops of exponent 4 are groups), which is false. So,  $A_\rho = (V, U, JWJ) \notin AUT(L, \cdot)$ . ♠

**Corollary 5.1** *In a C-loop  $(L, \cdot)$ , if  $(I, L_z^2, JL_z^2J) \in AUT(L)$ , and  $(R_z^2, I, JR_z^2J) \in AUT(L)$  for all  $z \in L$ , then the following are true :*

1.  $L$  is flexible.
2.  $(xy)^2 = (yx)^2$  for all  $x, y \in L$ .
3.  $x \mapsto x^3$  is an anti-automorphism.

**Proof**

This follows by Theorem 5.2, Lemma 5.1 and Corollary 5.2 of [15]. ♠

**Theorem 5.4** *A central square C-loop of exponent 4 is a group.*

**Proof**

To prove this, it shall be shown that the right inner mapping

$R(x, y) = I$  for all  $x, y \in L$ . Corollary 5.1 is used. Let  $w \in L$ .  
 $wR(x, y) = wR_xR_yR_{xy}^{-1} = (wx)y \cdot (xy)^{-1} = (wx)(x^2yx^2) \cdot (xy)^{-1} = (wx^3)(yx^2) \cdot (xy)^{-1} = (w^2(w^3x^3))(yx^2) \cdot (xy)^{-1} = (w^2(xw)^3)(yx^2) \cdot (xy)^{-1} = w^2(xw)^3 \cdot (yx^2)(xy)^{-1} = w^2(xw)^3 \cdot [y \cdot x^2(xy)^{-1}] = w^2(xw)^3 \cdot [y \cdot x^2(y^{-1}x^{-1})] = w^2(xw)^3 \cdot [y(y^{-1}x^{-1} \cdot x^2)] = w^2(xw)^3 \cdot [y(y^{-1}x)] = w^2(xw)^3 \cdot x = w^2(w^3x^3) \cdot x = w^2 \cdot (w^3x^3)x = w^2 \cdot (w^3x^{-1})x = w^2w^3 = w^5 = w$  if and only if  $R(x, y) = I$  if and only if  $R_xR_yR_{xy}^{-1} = I$  if and only if  $R_xR_y = R_{xy}$  if and only if  $zR_xR_y = zR_{xy}$  if and only if  $zx \cdot y = z \cdot xy$  if and only if  $L$  is a group. Hence the claim is true. ♠

**Corollary 5.2** *In a C-loop  $(L, \cdot)$ , if  $(I, L_z^2, JL_z^2J) \in AUT(L)$ , and  $(R_z^2, I, JR_z^2J) \in AUT(L)$  for all  $z \in L$ , then  $L$  is a group.*

**Proof**

This follows from Theorem 5.2 and Theorem 5.4. ♠

**Remark 5.2** *Central square C-loops of exponent 4 are A-loops.*

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